MYSTERIOUS PRECESSION OF MERCURY

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1. Introduction

Precession 1. This is the rotation of the plane of orbit.

This can be predicted to some accuracy using Newtonian Mechanics. However, for Mercury in particular, the results are imprecise. Einstiens theory predicts the precession accurately to 575" per century.

Principle of Equivalence 1. Locally General Relativity recognizes acceleration and gravity as equivalent.

2. Some Math

Notation 1.

\[ \partial_{x^i} = \vec{x}_i = \partial_u \vec{x} \]

Example 1. In \( \mathbb{R}^3 \) we have \( \partial_u \vec{x} = (\partial_u x_1, \partial_u x_2, \partial_u x_3) \).

Chain Rule 1. \[ = \partial_i \vec{x} = \sum x_i \dot{u}_i \]

It follows that the description for the 1-form is \( d\vec{x} = \sum x_i du^i \). The expression \( d\vec{x} \cdot d\vec{x} = \sum_{i,j} x_i x_j du^i du^j \) gives the metric. Since this is a quadratic form it can be represented as \( [g_{ij}] \).

Example 2. Parametrize We can parametrize \( S^2 \) using \( u_1 = \phi, u_2 \theta \) then we have

\[ \text{Figure 1. orbit of mercury displays precession} \]
\[= (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)\]

Now compute \(\partial x_i\). Since eventually we need \(g_{ij} = \partial x_i \cdot \partial x_j\).

\[
\partial_{u_1} \mathbf{x} = (\rho \cos \phi \cos \theta, \rho \cos \phi \sin \theta, -\rho \sin \phi) \\
\partial_{u_2} \mathbf{x} = (\sin \phi \sin \theta, \sin \phi \cos \theta, 0)
\]

Therefore,

\[
g_{ij} = \\
\begin{pmatrix}
1 & 0 \\
0 & \sin^2 \phi
\end{pmatrix}
\]

Some examples of metrics are

\[
\begin{align*}
ds^2 &= -dx^2 - dy^2 - dz^2 + c^2 dt^2 \\
d\tau^2 &= -dr^2 - r^2 d\phi^2 - r^2 \sin^2 \theta d\theta^2 + dt^2 \\
d\tau^2 &= -e^{\lambda(r)} dr^2 - r^2 (d\phi^2 + \sin^2 \phi d\theta^2) + e^{\nu(r)} dt^2
\end{align*}
\]

In the course of this presentation we will see quantities of form \(R_{ijkl}\), or \(R'_{ijkl}\), or \(\Gamma^i_{jk}\). To give a sense of concreteness think of \(R_{ijkl}\) as coming from the following matrix

\[
\begin{bmatrix}
[R_{i1k1}] & \cdots & [R_{i1km}] \\
\vdots & \ddots & \vdots \\
[R_{imk1}] & \cdots & [R_{imkm}]
\end{bmatrix}
\]

3. Important Equations

**Covariant Differentiation 1.** Let \(X^i_j\) denote the symbol for covariant differentiation, \(X^i_j\) the partial derivative, and \(\Gamma^i_{jk}\) denote the christoffel symbol.
Figure 2. Since orthogonal projection onto tangents gives zero connection.

Figure 3. another vector field on $S^1$

Figure 4. angle difference after parallel transport on closed curve corresponds to curvature

The connection is given by

$$X^i_{\ j} = X^i_{\ j} + \sum_k \Gamma^i_{\ jk} X^k$$

(2)

The Christoffel symbol is related to the metric through the relation

$$\Gamma^i_{\ jk} = \frac{1}{2} \sum_l g^{il} (\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk})$$

(3)

Remark 1. $g^{il}$ comes out of the relation $[g^{il}] = [g_{il}]^{-1}$.

Other important formulas are

$$R^i_{\ jkl} = -\partial_l \Gamma^i_{\ jk} + \partial_k \Gamma^i_{\ jl} + \sum_h (-\Gamma^h_{\ jk} \Gamma^i_{\ hl} + \Gamma^h_{\ jl} \Gamma^i_{\ hk})$$

$$R_{jl} = \sum_i R^i_{\ jil}$$

(4)
4. GEODESICS

**Geodesic 1.** let $\kappa$ denote the curvature of a curve $\rho$ on a surface then $\rho$ is a *Geodesic* if its projection onto a copy of the tangent space is zero $\text{Proj} \kappa = 0$

When a curve is given a parametrization in arc length geodesics satisfy the differential equation

$$0 = \partial_t^2 u^i + \sum_{j,k} \Gamma^i_{jk} \partial_t u^j \partial_t u^k.\quad (5)$$

5. PHYSICAL CONSIDERATIONS

Gravity in Einstein’s theory is built into the geometry of space-time and does not arise in the field equation. Furthermore, the einstein tensor in free space vanishes

$$C^i_k = g^{ij} R_{jk} - \frac{1}{2} R \delta^i_k = 0.\quad (6)$$

6. COMPUTING PRECESSION OF MERCURY

The metric

$$d \tau^2 = -e^{\lambda(r)} dr^2 - r^2 (d\phi^2 + \sin \phi d\theta^2) + e^{\nu(r)} dt^2$$

features prominently in this problem. Through using the Einstein tensor, and that in the clasical limit the Newtonian theory should agree with the Relativistic theory gives the *Swarzchild* metric
\[ d\tau^2 = -(1 - \frac{2GM}{r})dr^2 - r^2(d\phi^2 + \sin \phi d\theta^2) + (1 - \frac{2GM}{r})dt^2 \tag{7} \]

Through some computation the Christoffel symbols are given as follows

\[
\begin{align*}
\Gamma^1_{11} &= -\frac{GM}{r^2-2GMr} & \Gamma^2_{12} &= \Gamma^3_{13} = \frac{1}{r} \\
\Gamma^1_{22} &= -r(1 - \frac{2GM}{r}) & \Gamma^2_{33} &= -\sin \phi \cos \phi \\
\Gamma^1_{33} &= (2GM - r) \sin^2 \phi & \Gamma^3_{33} &= \cot \phi \\
\Gamma^1_{44} &= \frac{(1 - \frac{2GM}{r})(2GM)}{2} & \Gamma^4_{14} &= \frac{GM}{r^2-2GMr}
\end{align*}
\]

Therefore the geodesic equation \(0 = \frac{\partial^2}{\partial t^2}u^i + \sum_{j,k} \Gamma^i_{jk} \frac{\partial}{\partial t}u^j \frac{\partial}{\partial t}u^k\) has only 4 differential equations in it. We shall consider only 3 of them and \(d\tau^2 = g_{ij}dx^idx^j\)

\[
\begin{align*}
(1) & \quad -\frac{1}{1 - \frac{2GM}{r}}(\partial_r r)^2 - r^2(\partial_r \phi)^2 - r^2 \sin^2 \phi (\partial_r \theta)^2 + (1 - \frac{2GM}{r})(\partial_r t)^2 = 1 \\
(2) & \quad \partial_r^2 \phi + \frac{2}{r} \partial_r r \partial_r \phi - \sin \phi \cos \phi (\partial_r \theta)^2 = 0 \\
(3) & \quad \partial_r^2 \theta + \frac{r}{\sin \phi} \partial_r r \partial_r \theta + 2 \cot \phi \partial_r \phi \partial_r \theta = 0 \\
(4) & \quad \partial_r^2 t + \frac{2GM}{r^2-2GMr} \partial_r r \partial_r t = 0
\end{align*}
\]

By considering the initial condition we recognize that initially an orbit must start in a plane. Therefore we can assume \(\partial_r \phi, \cos \phi = 1\). But equation (2) forces this to be the case for all time. Hence \(\phi = \frac{\pi}{2}\).

The equations simplify further

\[
\begin{align*}
(1) & \quad -\frac{1}{1 - \frac{2GM}{r}}(\partial_r r)^2 - r^2(\partial_r \phi)^2 + (1 - \frac{2GM}{r})(\partial_r t)^2 = 1 \\
(2) & \quad \partial_r^2 \theta + \frac{2}{r} \partial_r r \partial_r \theta = 0 \\
(3) & \quad \partial_r^2 t + \frac{2GM}{r^2-2GMr} \partial_r r \partial_r t = 0
\end{align*}
\]

Multiplying 2 by \(r^2\) and 3 by \((1 - \frac{2GM}{r})\) we can solve 2 and 3 to get

\[
\begin{align*}
\partial_r \theta &= h \\
1 - 2GM &\frac{1}{r\partial_r t} = \beta
\end{align*}
\]

Now 1 becomes

\[
\frac{r^2 \partial_r \theta}{1 - 2GM \frac{1}{r\partial_r t} = \beta}
\]

\[(8)\]
\[ \frac{-1}{r^4} (\partial_\theta r)^2 - \left( \frac{GM}{r^2} \right) \left( \frac{\beta}{h} \right)^2 = \frac{2GM}{h^2} \tag{9} \]

making substitution and relabelling constants we can get

\[ (\partial_\theta u)^2 = 2GM \left( u^3 - \frac{u^2}{2GM} + \beta_1 u + \beta_0 \right) \]

\[ \Rightarrow (\partial_\theta u)^2 = 2GM (u - u_1)(u - u_2)(u - \frac{1}{2GM} + u_1 + u) \]

hence \[ \partial_u \theta = \frac{1}{\sqrt{(u-u_1)(u-u_2)(1-2GM(u+u_1+u_2))}} \]

\[ \approx \frac{1+GM(u+u_1+u_2)}{\sqrt{(u-u_1)(u-u_2)}} \tag{10} \]

The first approximation of the orbit give \[ u = \frac{1+e \cos \theta}{l} \]. If we compute the integral for 1 revolution we get a quantity \[ 2\pi + 6\pi \frac{GM}{a(1-e^2)} \]. The extra term is the precession \( \times T \). Using

\[ G = 6.67 \times 10^{-11} \frac{m^3}{kg \cdot sec^2} \]

\[ c = 3.0 \times 10^8 \frac{m}{sec} \]

\[ a = \text{mean distance from Mercury} \]

\[ e = \text{eccentricity}=0.206 \]

\[ T = \text{period}=88 \text{days} \]

\[ \implies \text{precession} = 43.1^\circ/\text{century} \tag{11} \]