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Energy loss due to binary collisions in a relativistic plasma

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Energy-transfer equations for a test particle in a fully ionized plasma are presented. The charged particles interact via Lienard-Wiechert potentials and the dominant contribution to the scatterings are from small-angle binary collisions. Asymptotic expansions of the energy rate equations are presented for all cases where either the test or the field particles or both are relativistic. These asymptotic equations are then used to derive equations determining rates of equipartition of energy, from which appropriate equipartition times are deduced. This work represents a relativistic generalization of nonrelativistic binary-collision Coulomb scattering.

I. INTRODUCTION

The study of the rate of energy loss resulting from a successive series of binary collisions between particles interacting via an inverse-square-law force has a long and outstanding history. Landau[1] first calculated the rate of energy loss of a charged test particle in a fully ionized nonrelativistic plasma. Since then this result has been reproduced by many authors, notably Spitzer,[2] Longmire,[3] along with Butler and Buckingham.[4] In all of these studies the Coulomb logarithm factor $\ln\Lambda$, arising from the predominantly large number of small-angle scattering events that occur in Coulomb scattering, was treated as a constant. One of us[5] studied in detail the resulting structure of the energy-loss equations when the correct velocity dependence of the $\ln\Lambda$ factor is taken into account. In the corresponding gravitational case, Chandrasekar[6] derived the same energy-transfer equations in his classic studies of stellar dynamics with the only difference in the results being that the Coulomb coupling constant is replaced by the gravitational coupling constant.

The discussion of Coulomb binary collisions presented in these studies was confined to a nonrelativistic treatment of the problem. This paper presents the relativistic generalization of Coulomb binary collisions in a fully ionized plasma. We reformulate the entire problem in a relativistic context. The rate of energy loss of a test particle is calculated in the general case where the test particle and/or the field (plasma) particle are relativistic. The exact expression for the rate of energy loss is obtained. We then study various physically interesting regions where the appropriate asymptotic expansions can be made.

The relativistic rate of energy transfer is of importance for a relativistic plasma for the rate of binary-collision energy loss determines the efficiency of heating the plasma by means of elastic scatterings. Furthermore, under relativistic conditions all characteristic relaxation times and other important quantities that relate to energy and momentum transfer must be calculated relativistically.

The relativistic calculations presented here have relevant physical applications for plasma in the laboratory as well as in astrophysics. In particular, for thermonuclear plasma, where the temperature $T \gtrsim 10^8$ °K, the electrons must be considered relativistic. Currently, in laboratory plasmas very intense electron beams are being employed[7] to heat a fully ionized plasma—hopefully to thermonuclear temperatures. While the coupling between such relativistic electron beams and plasmas of thermonuclear interest via binary collisions is relatively weak compared to certain collective interactions, there is nonetheless some binary-collision heating. Also, the characteristic energy relaxation times here must be obtained from relativistic calculations such as those given in this paper.

There are also many problems in plasma astrophysics where the charged-particle scatterings should be treated relativistically. Relativistic charged particles and their interaction with a plasma are important in explaining various phenomena associated with galactic nuclei,[8] radio galaxies,[8] quasars,[8] and pulsars.[8,9] The propagation of cosmic rays in intergalactic space along with the specific questions concerning the detailed intrinsic structure of the cosmic rays themselves are further examples of important problems in relativistic plasma astrophysics.[10]

Besides the practical value for plasma physics, the calculations given here are of importance in relativistic kinetic theory. In the nonrelativistic case, Montgomery and Tidman[11] have shown how the energy-rate equations can be obtained from the Fokker-Planck equation and May[12] has shown how these results follow directly from the Boltzmann equation. In the sections that follow we will find that virtually all of the quantities that were invariant in the nonrelativistic treatment cease to be so in the relativistic formulation. Much care will be taken to show how the various physical quantities that enter into the calculation transform. This work is therefore important for kinetic theory since any calculation of the rate of energy transfer using a relativistic kinetic equation must reproduce the result obtained in this paper.

We present the calculations necessary to obtain the energy-transfer equations as well as study their properties in Secs. II–VII. In Sec. VIII we give a discussion of our results and return to discuss briefly the present level of development of appropriate relativistic kinetic equations for a plasma.

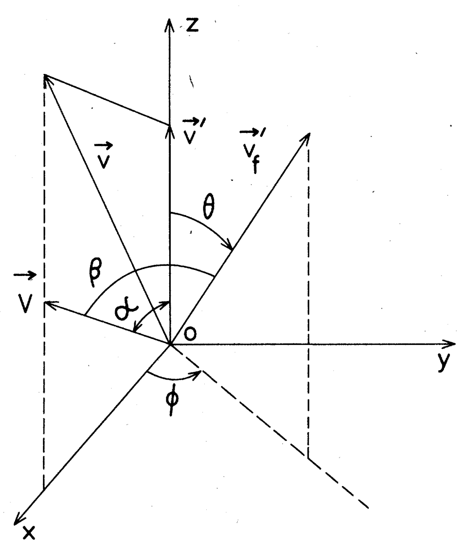

FIG. 1. Coordinate system for calculating the energy-transfer expression (2.4).

II. ENERGY CHANGE IN A RELATIVISTIC COLLISION

We calculate here the change in energy of a test particle having momentum $\vec{p}$ with a field particle having laboratory-system (l.s.) momentum $\vec{q}$ before the collision. The calculation is most readily carried out in the center-of-mass-system (c.m.s.) and the appropriate quantities later transformed back to the l.s. l.s. denotes the laboratory frame, and c.m.s. the center-of-mass frame.

l.s. quantities are unprimed and c.m.s. quantities are primed. Unsubscripted quantities refer to values before collision and the subscript f indicates final values.

Let $\vec{V}$ be the velocity of the c.m.; then the transformations from l.s. to c.m.s. are given by

$$\begin{pmatrix} p_x \\ p_y \\ p_z \\ E(\vec{p}) \end{pmatrix} = \begin{pmatrix} \Gamma\left(p_x' + \frac{V}{c^2}E(\vec{p}\,')\right) \\ p_y' \\ p_z' \\ \Gamma[E(\vec{p}\,') + Vp_x'] \end{pmatrix}, \quad \Gamma = \left(1 - \frac{V^2}{c^2}\right)^{-1/2} \tag{2.1}$$

$$\vec{V} = \frac{(\vec{p}+\vec{q})c^2}{E(\vec{p}) + E(\vec{q})}, \quad E(\vec{p}) = (p^2c^2 + M^2c^4)^{1/2},$$

where M is the rest mass of the test particle and m will denote the rest mass of the field particle.

Conservation of momentum requires that both $\vec{V}$ and $\vec{p}\,' + \vec{q}\,' = 0$ are invariants. Conservation of energy requires that both $E(p')$ and $E(q')$ are invariants.

The change of energy of the test particle is

$$\begin{aligned} \Delta E &= E(p_f) - E(p) \\ &= V[(p_x')_f - (p_x')]\Gamma \\ &= \vec{V} \cdot (\vec{p}_f\,' - \vec{p}\,')\Gamma . \end{aligned} \tag{2.2}$$

Since the scattering takes place in a plane in the c.m.s., we may relate all scattering angles to the plane defined by the initial velocities $\vec{v}$ and $\vec{w}$ as shown in Fig. 1. In Fig. 1 the z axis has been taken as the direction of $\vec{v}\,'$ and the zx plane is the plane determined by $\vec{v}$ and $\vec{w}$. We note at this point some simple definitions; for the l.s.,

$$\begin{aligned} &\vec{p} = M\gamma_1\vec{v}, \quad \vec{q} = m\gamma_2\vec{w}, \\ &\gamma_1 = (1 - v^2/c^2)^{-1/2}, \quad \gamma_2 = (1 - w^2/c^2)^{-1/2}, \end{aligned} \tag{2.3}$$

and similarly we have Eq. (2.3) again with all quantities primed for the c.m.s. The subscript 1 will henceforth denote the test particle and the subscript 2 the field particle. Thus, using Fig. 1 and Eq. (2.2), after a straightforward calculation we find that

$$\Delta E = -\Gamma[2\sin^2\theta/2(\vec{V} \cdot p') - \sin\theta\cos\phi\,|\vec{V} \times \vec{p}\,'|]. \tag{2.4}$$

Equation (2.4) is the exact relativistic generalization of the comparable result for the nonrelativistic case.[3,4] The angles θ and ϕ are c.m.s. scattering angles. We do not transform them back to l.s. values as we will average over them directly later. The test particle momentum $\vec{p}'$ can be transformed back to the l.s. value by the inverse of the transformation given in Eq. (2.1).

III. KINEMATIC RESULTS

In much of the work to come we shall have to transform c.m.s. values back to their l.s. values. The results needed to do this are readily obtained from the velocity transformations and are as follows:

For the transformation of the Lorentz factors,

$$\gamma_1' = (1 - \vec{V}\cdot\vec{v}/c^2)\Gamma\gamma_1, \quad \gamma_2' = (1 - \vec{V}\cdot\vec{w}/c^2)\Gamma\gamma_2,$$

$$\Gamma = (1 - \vec{V}\cdot\vec{V}/c^2)^{-1/2}, \quad \vec{V} = \frac{M\gamma_1\vec{v} + m\gamma_2\vec{w}}{M\gamma_1 + m\gamma_2}, \tag{3.1}$$

from the definition of V in Eq. (2.1),

$$1 - \frac{\vec{V}\cdot\vec{V}}{c^2} = \frac{1 + (m/M)^2 + 2(m/M)\gamma_1\gamma_2(1 - \vec{v}\cdot\vec{w}/c^2)}{[\gamma_1 + (m/M)\gamma_2]^2},$$

$$1 - \frac{\vec{V}\cdot\vec{v}}{c^2} = \frac{1 + (m/M)\gamma_1\gamma_2(1 - \vec{v}\cdot\vec{w}/c^2)}{\gamma_1[\gamma_1 + (m/M)\gamma_2]}, \tag{3.2}$$

$$1 - \frac{\vec{V}\cdot\vec{w}}{c^2} = \frac{1 + (M/m)\gamma_1\gamma_2(1 - \vec{v}\cdot\vec{w}/c^2)}{\gamma_2[\gamma_2 + (M/m)\gamma_1]}.$$

We also note that conservation of momentum gives

$$M\gamma_1' v' = m\gamma_2' w',$$

or, equivalently, (3.3)

$$M^2[(\gamma_1')^2 - 1] = m^2[(\gamma_2')^2 - 1].$$

IV. RELATIVISTIC COULOMB CROSS SECTION

We will now calculate the relativistic Coulomb scattering cross section. For this purpose the trajectories of the charged particles must be calculated and then, knowing these, the computation of the cross section follows. This is readily done using Newton's equations of motion in the nonrelativistic case and leads directly to the standard Rutherford cross section employed in Refs. 1–5. For the relativistic case this calculation is much more difficult.

Newton's equations of motion are quite difficult to solve now. It turns out to be more advantageous to calculate the trajectories from the Hamilton-Jacobi equation. Finally, the calculation is made even harder because of the question of the nature of the relativistic interaction (we shall return again to this point in Sec. VIII). It is assumed that the relativistic interaction is characterized by the Lienard-Wiechert potentials.

The Lienard-Wiechert potentials are appropriate for charged particles traveling with uniform motion. As the dominant contribution to the scatterings are from small-angle binary collisions, we expect the charged particles to be deflected slightly, thus changing their directions but leaving their speeds essentially unchanged. The Lienard-Wiechert fields are therefore a very good choice for the appropriate relativistic interaction because of their long-ranged, inverse-square-law nature. They are the relativistic generalizations of the Coulomb force field, suitable for use in the nonrelativistic case.

We have not attempted to derive the exact scattering cross section for the potentials but give here the derivation of the scattering cross section in the impulse approximation, for particles moving under the action of the Lienard-Wiechert potentials. Owing to the long-range nature of this interaction, virtually all of the energy-transfer results from a very large number of very-small-angle scatterings. It is just this situation for which the impulse approximation is ideal. Thus the very nature of the dominant contribution resulting from these very-small-angle scattering events makes possible both the choice of the Lienard-Wiechert potentials as well as the use of the impulse approximation in carrying out these relativistic calculations.

We will then compare this cross section with a known exact cross section for relativistic charged particles when both the test and field particles are electrons. This cross section, the Møller[13] cross section, reduces to the result we obtain from the impulse approximation. Furthermore, since it is defined for all scattering angles, we can then immediately deduce the nature of the relativistic Coulomb cross section for large angles.

Thus we can use our approximate cross section as confidently as if the exact result were known for all scattering angles. We now give the derivation in the impulse approximation and then make the detailed comparison with the case of the Møller cross section.

The calculation is performed in the c.m.s. as Eq. (2.4) is expressed directly in terms of c.m.s. angles. As shown in Fig. 2, the scattering is in a plane in the c.m.s.

The impact parameter is b and $\vec{r}_0$ is the radius vector joining the test particle of charge Ze and the field particle of charge ze. The unit vectors $\vec{i}$ and $\vec{j}$ are vectors respectively parallel and perpendicular to the direction of motion of the particles.

The fields produced by the test particle are given by

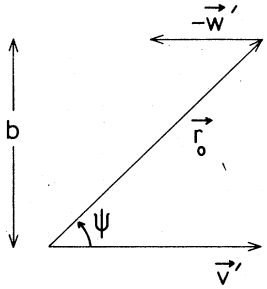

FIG. 2. Coordinate system for calculating the scattering cross section.

$$\vec{E} = \frac{Ze\vec{r}_0}{r^3}\left(1 - \frac{(v')^2}{c^2}\right),$$
$$\vec{B} = \frac{\vec{v}'}{c} \times \vec{E}, \qquad (4.1)$$
$$r = r_0[1 - (v')^2/c^2 \sin^2\psi]^{1/2}.$$

The force on the field particle is then

$$\vec{F} = ze[\vec{E} + (\vec{w}'/c) \times \vec{B}]$$
$$= ze[\vec{E}(1 + v'w'/c^2) - (v'w'/c^2)E_x\vec{i}]. \qquad (4.2)$$

In the impulse approximation the only component of force that enters the calculation is $F_y \equiv \vec{F} \cdot \vec{j}$. We therefore have

$$F_y = \frac{(Zze^2)b[1 - (v')^2/c^2][1 + v'w'/c^2]}{[r_0^2 - b^2(v')^2/c^2]^{3/2}}, \qquad (4.3)$$

where $\sin\psi = b/r_0$ throughout. Also,

$$r_0^2 = b^2 + (x_1 - x_2)^2$$
$$= b^2 + (v' + w')^2 t^2, \qquad (4.4)$$

where $x_1 = v't$, $x_2 = -w't$, and t is time. For the total impulse produced by the force F_y,

$$\int_{-\infty}^{+\infty} F_y\, dt = \Delta p' = \frac{2(Zze^2)(1 + v'w'/c^2)}{v' + w'}\,\frac{1}{b}. \qquad (4.5)$$

The c.m.s. scattering angle θ is therefore

$$\theta = \frac{\Delta p'}{p'} = \frac{2(Zze^2)(1 + v'w'/c^2)c^2}{E_1'v'(v' + w')}\,\frac{1}{b}, \qquad (4.6)$$

where $p' = v'E_1'/c^2$ and $E_1' = Mc^2\gamma_1'$.

The differential scattering cross section is then

$$\frac{d\sigma}{d\Omega} = -\frac{b}{\sin\theta}\,\frac{db}{d\theta} = \left(\frac{2(Zze^2)c^2(1 + v'w'/c^2)}{E_1'v'(v' + w')}\right)^2 \frac{1}{\theta^4}. \qquad (4.7)$$

We note also, for later use, that in the nonrelativistic limit Eq. (4.7) reduces to

$$\frac{d\sigma}{d\Omega} = \frac{4(Zze^2)^2}{\mu^2 g^4}\,\frac{1}{\theta^4},$$

where $\mu = mM/(M + m)$ and $g = |\vec{v} - \vec{w}|$. This is identical to the Rutherford result in the small-angle approximation. We see from Eq. (4.7) that the cross section is in terms of c.m.s. speeds, which will have to be transformed back to l.s. values using the results in Sec. III. This differs from the nonrelativistic value which is expressed in terms of the nonrelativistic invariant g.

We now compare our final result (4.7) with a known relativistic result, the Møller cross section. The Møller result is a relativistic quantum-mechanical calculation of electron-electron scattering which combines the Lienard-Wiechert potentials with the quantum effects of exchange and spin. The Møller result expressed in terms of c.m.s. values is

$$\frac{d\sigma}{d\Omega} = \frac{R_0^2 A(\theta, \phi)}{(\gamma')^2(2\beta'\gamma')^4 \sin^4\frac{1}{2}\theta},$$

$$A(\theta, \phi) = \tfrac{1}{2}\{[2(\gamma')^2 - 1] + (\gamma')^4[1 + (\beta')^2\cos\theta]^2 - 2(\gamma')^2[1 - (\beta')^2\cos\theta] + 2\}, \qquad (4.8)$$

where R_0 is the classical radius of the electron and $\beta' = v'/c$ [remembering that in the c.m.s. $v' = w'$ for equal-mass particles as seen from Eq. (3.3)]. The cross section (4.8) is the Møller result without inclusion of the exchange-scattering contribution, this having no classical counterpart. The Møller cross section in Eq. (4.8) is thus given only for direct scattering; in other words, we include the scattering shown in Fig. 3 and omit the exchange diagram.

The denominator of Eq. (4.8) has the standard Coulomb form and the numerator $A(\theta, \gamma')$ includes the effects of both relativity and spin. We expect that in the forward-angle (small-angle) region the spin contribution should disappear and that Eq. (4.8) should reduce to our classical result (4.7). This is indeed the case. Because the relativistic factors in the denominator of Eqs. (4.8) and (4.7) are identical and those in the numerator are identical in the small-angle region, where the spin contribution is no longer present, we surmise that Eq. (4.7) would be the exact classical relativistic

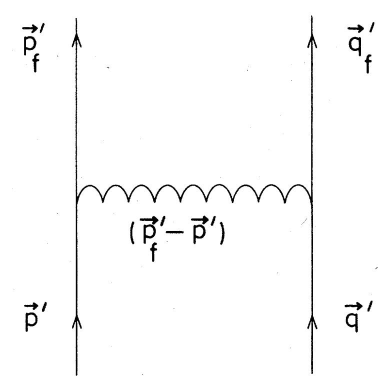

FIG. 3. Direct scattering diagram for electron-electron scattering contribution to the Møller cross section.

result if θ^4 in Eq. (4.7) were replaced by $(2\sin\frac{1}{2}\theta)^4$. An exact calculation is of course desirable, but this comparison of Eq. (4.7) with Eq. (4.8) indicates that the impulse approximation has included the relativistic factors exactly and simply given the small-angle approximation for the factor $(2\sin\frac{1}{2}\theta)^4$. We note further that the replacement of θ^4 by $(2\sin\frac{1}{2}\theta)^4$ in the nonrelativistic limit of Eq. (4.7) produces the exact Rutherford result.

V. RATE OF ENERGY LOSS

We are now ready to calculate the total rate of energy loss of a test particle to a plasma. The procedure is the same as that in Refs. 1–5; we first calculate the average change of energy in a single collision and then integrate over all the collisions the test particle has with the field particles of the plasma.

We use Eqs. (2.4) and (4.7) to calculate the average energy change on a single collision,

$$\Delta E = \frac{1}{\sigma_t'}\int \Delta E \frac{d\sigma}{d\Omega} d\Omega\,, \qquad (5.1)$$

where σ_t' is the total scattering cross section in the c.m.s.; since Eq. (4.7) is a small-angle-approximation result, we must make the same small-angle approximation to Eq. (2.4), so obtaining for Eq. (5.1)

$$\langle\Delta E\rangle = \frac{-4\pi(Zze^2)^2c^4(1+v'w'/c^2)^2(\vec{V}\cdot\vec{p}')\Gamma}{\sigma_t'(E_1')^2[v'(v'+w')]^2}\int\frac{d\theta}{\theta}. \qquad (5.2)$$

The integration over θ diverges at the small-angle end (just as in the nonrelativistic case) and must be terminated at a cutoff angle θ_0 corresponding to a maximum impact parameter $b_{\max}$. Equation (4.6) gives the relationship between θ_0 and $b_{\max}$.

We now write for the θ integral in Eq. (5.2)

$$\int\frac{d\theta}{\theta} = \ln\frac{2}{\theta_0} \equiv \ln\Lambda\,, \qquad (5.3)$$

where Eq. (4.6) gives the detailed expression for θ_0. We have adopted the argument of the logarithm in Eq. (5.3) for the following reasons. The Møller cross section has been used in Eq. (5.1) and gives Eq. (5.2) exactly with $\ln(\sin\frac{1}{2}\theta_0)^{-1} \simeq \ln(2/\theta_0) \equiv \ln\Lambda$ in place of the θ integral. There is also an additional term not proportional to $\ln\Lambda$ and small everywhere (from the nonrelativistic to the extreme relativistic limit). This term can be traced to the effect of spin [included in Eq. (4.8)] and would not appear classically. We have calculated Eq. (5.1) with Eq. (4.7) and θ^4 replaced by $[2\sin\frac{1}{2}\theta]^4$ and obtained exactly Eq. (5.2) with $\ln(2/\theta_0)$ appearing in place of integration. All these results are understandable in the light of the cross-section comparison discussed in Sec. IV. Thus we find Eqs. (5.2) and (5.3) for our final result.

To obtain the total rate of energy loss we must now integrate over all collisions and thus need the relativistic form of $d\nu$, the collision frequency. Following Landau and Lifshitz,[14] we find for $d\nu$ the number of collisions between a test particle and a field particle with momentum between $\vec{q}$ and $\vec{q}+d\vec{q}$ in the volume dV in the time dt in the l.s.,

$$d\nu = \sigma_t|\vec{v}-\vec{w}|\rho_1\rho_2 f(\vec{q})d\vec{q}\,dV\,dt\,. \qquad (5.4)$$

While $d\nu$ given in Eq. (5.4) is an invariant, not each of the quantities that make it up is. This is in contrast to the nonrelativistic situation, where $d\nu$ and all its factors are invariant quantities. The density of field particles in the l.s. is denoted by ρ_2 and the density of test particles in the l.s. by ρ_1. The total l.s. cross section is σ_t and $f(\vec{q})$ is the field-particle distribution function given explicitly in terms of the field particle's l.s. momentum. While the distribution function $f(\vec{q})$ is not an invariant, the quantity $f(\vec{q})d\vec{q}$ is and must be normalized so that $\int f(\vec{q})d\vec{q}=1$. [Note that $f(\vec{w})\,d\vec{w}$ is not a relativistic invariant.]

Equation (5.2) contains σ_t' and relativistically $\sigma_t'\neq\sigma_t$. We must therefore transform Eq. (5.4) so that it contains σ_t' and gives all other quantities in terms of their l.s. values. Using the Pauli invariant as given by Landau and Lifshitz,[14]

$$\sigma_t|\vec{v}-\vec{w}|/(1-\vec{v}\cdot\vec{w}/c^2) = \text{invariant}\,, \qquad (5.5)$$

we readily find for Eq. (5.4)

$$d\nu = \sigma_t'[(\vec{v}-\vec{w})^2-(\vec{v}\times\vec{w})^2/c^2]^{1/2}\rho_1\rho_2 f(\vec{q})d\vec{q}\,dV\,dt\,. \qquad (5.6)$$

Since we are dealing with only one test particle, we have $\int\rho_1 dV=1$. If we were dealing with N test particles in the l.s., then $\int\rho_1 dV=N$. We drop the subscript 2 in ρ_2 in the future and write it simply as ρ, the density of field particles in the l.s.

Using Eqs. (5.2), (5.3), and (5.6) we now have for the total rate of energy loss in the l.s.

$$\frac{dE}{dt} = -\int\frac{4\pi(Zze^2)^2c^4(1+v'w'/c^2)^2(\vec{V}\cdot\vec{p}')\Gamma\rho\ln\Lambda}{(E_1')^2[v'(v'+w')]^2}\times[(\vec{v}-\vec{w})^2-(\vec{v}\times\vec{w})^2/c^2]^{1/2}f(\vec{q})d\vec{q}\,. \qquad (5.7)$$

Equation (5.7) still has many of its factors in terms of the c.m.s. To complete the calculation we transform all the remaining c.m.s. quantities to their l.s. values. The transformation inverse to Eq. (2.1) and all the transformations and identities given in Sec. III are used for this purpose. After a great deal of transformation one finds for the total rate of energy loss for a test particle in a plasma, with all quantities given in terms of their l.s. values,

$$\frac{dE}{dt}=\frac{-4\pi(Zze^2)^2\rho}{mc^2}\int\frac{\ln\Lambda[\gamma_1\gamma_2(1-\vec{v}\cdot\vec{w}/c^2)]^2\{(m/M)(\gamma_1-\gamma_2)+\gamma_1\gamma_2(1-\vec{v}\cdot\vec{w}/c^2)[\gamma_1-(m/M)\gamma_2]\}}{[1-\gamma_1^2\gamma_2^2(1-\vec{v}\cdot\vec{w}/c^2)^2]^2}\times[(\vec{v}-\vec{w})^2-(\vec{v}\times\vec{w})^2/c^2]^{1/2}f(\vec{q})d\vec{q}\,, \tag{5.8}$$

with γ_1 and γ_2 given by Eq. (2.3).

Upon taking the nonrelativistic limit, we readily obtain the exact result given in Refs. 1–4. Furthermore, using Eq. (4.6), we now find that $2/\theta_0 = \mu g^2 b_{\max}/Zze^2$, which is the precise value given for the nonrelativistic cutoff in Ref. 5. The maximum impact parameter introduced in the nonrelativistic treatment was the Debye length λ_D. We do not specify a value for $b_{\max}$ in the relativistic treatment but simply note that, owing to the long-range nature of the Coulomb potential, even in the relativistic study a cutoff θ_0 is required. We discuss this point again in Sec. VIII.

We now proceed to perform the integration over the orientation angle between the test and field particles in Eq. (5.8) using $\vec{v}\cdot\vec{w}=vw\cos\theta$ and $\vec{v}\times\vec{w}=vw\sin\theta$. The plasma distribution function is taken to be the isotropic relativistic Maxwellian, so that now $d\vec{q}=2\pi q^2\sin\theta\, d\theta\, dq$. It is at this stage that we make the approximation $\ln\Lambda$ a constant. This approximation is justified for the following reasons. Firstly, we are certainly not formally overlooking the velocity dependence of θ_0 as given by Eq. (4.6). Upon transforming Eq. (4.6) back into l.s. values we get a quantity which dependes strongly upon the orientation angle between $\vec{v}$ and $\vec{w}$. When this quantity is inserted into $\ln\Lambda$, noting that the rest of the integrand depends on θ in a nontrivial fashion, the integration over the orientation angle θ is even more formidable. We have not succeeded in doing this integration. The less formidable integration over θ, with $\ln\Lambda$ taken as a suitably chosen constant, can be done. Secondly, this is the first detailed study of the problem and it is reasonable to obtain results and study them in the constant $\ln\Lambda$ approximation. Of course, dominant and nondominant terms would arise from keeping the correct velocity dependence of θ_0, the precise value being given by Eq. (4.6) appropriately transformed, just as dominant and nondominant terms resulted in the nonrelativistic study when the correct velocity dependence of $\ln\Lambda$ was used.[5]

Now, to perform the integration over orientation angles, we make the change of variable

$$t=\gamma_1\gamma_2(1-\vec{v}\cdot\vec{w}/c^2)\,,\qquad dt=\gamma_1\gamma_2(vw/c^2)\sin\theta\, d\theta\,. \tag{5.9}$$

The entire integration over orientation angles can now be done using the indefinite integrals

$$\int\frac{t^2\,dt}{(t^2-1)^{3/2}}=-\frac{t}{(t^2-1)^{1/2}}+\ln\left|t+(t^2-1)^{1/2}\right|\,,$$

$$\int\frac{t^3\,dt}{(t^2-1)^{3/2}}=\frac{t^2-2}{(t^2-1)^{1/2}}\,.$$

The final result for Eq. (5.8), the total rate of energy loss, is

$$\frac{dE}{dt}=-\frac{8\pi^2c^2(Zze^2)^2\rho\ln\Lambda}{mc}\int\frac{F(v,w)}{vw(\gamma_1\gamma_2)^2}q^2f(q)dq\,, \tag{5.10}$$

where

$$F(v,w)=\left(\frac{m}{M}\gamma_1-\gamma_2\right)\left(\frac{-\gamma_1\gamma_2(1+vw/c^2)}{\{[\gamma_1\gamma_2(1+vw/c^2)]^2-1\}^{1/2}}+\frac{\gamma_1\gamma_2(1-vw/c^2)}{\{[\gamma_1\gamma_2(1-vw/c^2)]^2-1\}^{1/2}}+\ln\left|\frac{\gamma_1\gamma_2(1+vw/c^2)+\{[\gamma_1\gamma_2(1+vw/c^2)]^2-1\}^{1/2}}{\gamma_1\gamma_2(1-vw/c^2)+\{[\gamma_1\gamma_2(1-vw/c^2)]^2-1\}^{1/2}}\right|\right)+\left(\gamma_1-\frac{m}{M}\gamma_2\right)\times\left(\frac{[\gamma_1\gamma_2(1+vw/c^2)]^2-2}{\{[\gamma_1\gamma_2(1+vw/c^2)]^2-1\}^{1/2}}-\frac{[\gamma_1\gamma_2(1-vw/c^2)]^2-2}{\{[\gamma_1\gamma_2(1-vw/c^2)]^2-1\}^{1/2}}\right). \tag{5.11}$$

It remains to specify the distribution function. As stated earlier, this will be the isotropic relativistic Maxwellian distribution

$$f(q)q^2dq=\frac{a}{4\pi(mc)^3K_2(a)}\exp\{-a[1+q^2/(mc)^2]^{1/2}\}q^2dq\,, \tag{5.12}$$

where $a=mc^2/kT$, T is the plasma temperature, $K_2(a)$ is the modified Bessel function of the second kind of order 2, and the distribution is normalized such that $\int f(\vec{q})d\vec{q}=1$, as required. It is useful to express Eq. (5.12) explicitly in terms of the field particles' speed rather than the momentum, since all quantities except the distribution are so given in Eq. (5.10). We find, using Eq. (2.3), that Eq. (5.12) now becomes

$$f(q)q^2\,dq=\frac{a}{4\pi c^3K_2(a)}e^{-a\gamma_2}w^2(\gamma_2)^5dw\,. \tag{5.13}$$

It is a simple matter, using the asymptotic expansion of $K_2(a)$, to see that Eq. (5.13) reduces to the nonrelativistic distribution used in Refs. 1–5.

Equations (5.10), (5.11), and (5.13) give the total rate of energy loss of a test particle in a plasma. It is straightforward to show that it reduces to the correct result in the nonrelativistic limit.[1-4] It is a general result for any plasma, whether or not the test particle and/or the field particles are relativistic.

We remark, in closing this section, that the complexity of the result gives an indication of how hard it would be to include the correct velocity dependence of $\ln\Lambda$. In general, the final integration over w in Eq. (5.10) has to be done numerically. There are some asymptotic regions that are amenable to study and we look at these in the next section.

VI. ASYMPTOTIC LIMITS

When either the test particle or the field particles are relativistic, we have, to second order in v/c and w/c,

$$F(v,w)=\frac{2vw}{c^2}\left[\left(\gamma_2(\gamma_1^2-1)-\frac{m}{M}\gamma_1(\gamma_2^2-1)\right)+\frac{1}{3}\frac{v^2w^2}{c^4}\left(\frac{m}{M}\gamma_1-\gamma_2\right)\right]. \tag{6.1}$$

We shall now use Eq. (6.1) in Eq. (5.10) and study the two physically interesting cases for which Eq. (6.1) is an accurate approximation to $F(v,w)$. We note that all terms to order $(w/c)^2$ or $(v/c)^2$ inside the square brackets in Eq. (6.1) must be retained. This ensures that the correct nonrelativistic limit for either the test particle or the field particle is obtained.

For an ultrarelativistic test particle ($\gamma_1\gg1$) in a nonrelativistic plasma ($a\gg1$, so that $\gamma_2\approx1+w^2/2c^2$) the nonrelativistic limit of Eq. (5.13) is required and we find

$$\frac{dE_1}{dt}=-\frac{4\pi(Zze^2)\rho\ln\Lambda}{mc}\left(1-\frac{2kT}{E_1}\right). \tag{6.2}$$

We have dropped all terms of order $1/a$ and smaller and $E_1=Mc^2\gamma_1$. Equation (6.2) gives a zero total rate of energy loss when $E_1=2kT$. This energy has an interesting physical interpretation. It is the harmonic mean between $\frac{3}{2}kT$ and $3kT$, the two characteristic average thermal energies per particle for the nonrelativistic and ultrarelativistic cases, respectively. When the test particle energy is such that not only is $E_1\gg Mc^2(\gamma_1\gg1)$ but also $E_1\gg2kT$, then the total energy loss saturates at the value

$$\frac{dE_1}{dt}=-\frac{4\pi(Zze^2)^2\rho\ln\Lambda}{mc}. \tag{6.3}$$

Equation (6.3) is identical in form to the asymptotic value given in the nonrelativistic case[5] with the exception that c, the speed of light, now appears in the denominator rather than the test particle's speed v. This is the relativistic saturation value for v, as might have been expected. Equation (6.2) also shows us that, even though the test particle is ultrarelativistic, if the energies are also ordered as $Mc^2\ll E_1\ll kT\ll mc^2$, then

$$\frac{dE_1}{dt}=\frac{8\pi(Zze^2)^2\rho kT\ln\Lambda}{mcE_1}. \tag{6.4}$$

Equation (6.4) implies that, even though the test particle is ultrarelativistic, it gains (not loses) energy from the field particles. This is the case of an ultrarelativistic electron whose energy is still much below $2kT$, which is itself much below the rest-mass energy of an ion (e.g., $m/M\gg1$).

We note at this point that the energy appears in the form of a $1/E_1$ factor in Eq. (6.2). This fact will be of importance in the discussion of equipartition rates presented in the following section.

For a nonrelativisitic test particle ($\gamma_1\approx1+v^2/2c^2$) in an ultrarelativistic plasma ($a\ll1$, so that $\gamma_2\gg1$), we find

$$\frac{dE_1}{dt}=-\frac{2\pi(Zze^2)^2\rho a\ln\Lambda}{mc}\left(\frac{2}{3}\frac{v^2}{c^2}-\frac{2}{a}\frac{m}{M}\right), \tag{6.5}$$

or equivalently (remembering that $a=mc^2/kT$)

$$\frac{dE_1}{dt}=-\frac{8\pi(Zze^2)^2\rho\ln\Lambda}{3Mc(kT)}(E_1-\tfrac{3}{2}kT). \tag{6.5a}$$

In Eq. (6.5a), the test-particle energy is now specifically given by $E_1=\frac{1}{2}Mv^2$. The rest masses have cancelled out in the derivation.

Equation (6.5a) gives a zero total rate of energy loss for a nonrelativistic test particle in an ultrarelativistic plasma when $E_1=\frac{1}{2}Mv^2=\frac{3}{2}kT$. This is the physically acceptable result for this case. There are now two special limits to Eq. (6.5) or equivalently (6.5a). These can be seen most readily from Eq. (6.5) initially and then from the equivalent result (6.5a) in the end.

When the energy ratio a satisfies the inequality $m/M\ll a\ll1$ (e.g., $mc^2\ll kT\ll Mc^2$), then the test particle loses energy to the plasma at a rate proportional to E_1,

$$\frac{dE_1}{dt}=-\frac{8\pi(Zze^2)^2\rho\ln\Lambda}{3MckT}E_1\,. \tag{6.6}$$

This situation ($m/M\ll1$) requires that the plasma field particle be an electron and the test particle be an ion.

When the energy ratio a satisfies the inequalities $a \lesssim m/M$ and $a \ll 1$ (e.g., $kT \gg mc^2$ and $kT > Mc^2$), then it is impossible for the test particle to lose energy to the field particles. The test particle will gain energy from the plasma at a constant rate given by

$$\frac{dE_1}{dt} = \frac{4\pi(Zze^2)^2\rho\ln\Lambda}{Mc}. \qquad (6.7)$$

Equation (6.7) is identical in form to Eq. (6.3), with m replaced by M and the minus sign replaced by a plus sign.

We note that, when either the test particle or the field particle is relativistic, it is always the rest mass of the nonrelativistic species which appears in the rate equations [e.g., Eqs. (6.3) and (6.7)]. This is understandable, for an ultrarelativistic particle behaves essentially as a zero-rest-mass particle. Thus its mass should not appear explicitly.

To see this point even more clearly we give the total rate of energy loss of an ultrarelativistic test particle ($\gamma_1 \gg 1$) in an ultrarelativistic plasma ($a \ll 1$, so that $\gamma_2 \gg 1$). For this case,

$$F(v,w) = 2\gamma_1\gamma_2\frac{vw}{c^2}\left(\gamma_1 - \frac{m}{M}\gamma_2\right) \qquad (6.8)$$

is an excellent approximation for $F(v,w)$. Now, using Eq. (6.8) with Eq. (5.13), we have for the total rate of energy loss

$$\frac{dE_1}{dt} = -4\pi(Zze^2)^2\rho c\ln\Lambda\left(\left\langle\frac{1}{E_2}\right\rangle - \frac{1}{E_1}\right), \qquad (6.9)$$

where

$$\left\langle\frac{1}{E_2}\right\rangle = \int\frac{1}{E_2}f(q)4\pi q^2\,dq\,. \qquad (6.10)$$

The integral in Eq. (6.10) can be done exactly using Eq. (5.13), with the result

$$\langle 1/E_2\rangle = (1/mc^2)K_1(a)/K_2(a)\,. \qquad (6.11)$$

For $a \ll 1$, we readily find, using the asymptotic expansion for the modified Bessel functions

$$K_\nu(x) \sim \tfrac{1}{2}(\nu-1)!/(\tfrac{1}{2}x)^\nu, \quad (x \ll 1,\quad \nu > 0),$$

$$K_0(x) \sim -\ln x\,, \quad (x \ll 1)\,, \qquad (6.12)$$

that Eq. (6.11) becomes

$$\langle 1/E_2\rangle \simeq 1/2kT\,. \qquad (6.13)$$

Thus the final result for Eq. (6.9) can be written as

$$\frac{dE_1}{dt} = -4\pi(Zze^2)^2\rho c\ln\Lambda\left(\frac{1}{2kT} - \frac{1}{E_1}\right). \qquad (6.14)$$

As stated above, there is now no explicit dependence on M or m; we also notice the characteristic $1/E_1$ term, which always appears for the case of an ultrarelativistic test particle. Equation (6.14) shows that the test particle loses or gains energy from the plasma depending on whether $E_1 > 2kT$ or $E_1 < 2kT$. In particular, the test particle loses energy at the constant rate

$$\frac{dE_1}{dt} = -\frac{2\pi(Zze^2)^2\rho c\ln\Lambda}{kT}, \qquad (6.15)$$

when $E_1 \gg 2kT$.

The case of a nonrelativistic test particle in a nonrelativistic plasma was, of course, the subject of the studies in Refs. 1–5. This completes the study of all four appropriate asymptotic limits. There is still a wealth of information contained in our final result (5.10). It is always possible, in principle, to obtain expansions up to any power in $(w/c)^2$ or $(v/c)^2$. These results can generally be written in terms of a series of Bessel functions which can then be appropriately expanded in the various asymptotic limits.

VII. RATE OF EQUIPARTITION OF ENERGY

If the test particle is taken to be an ion and the field particles are electrons, then the rate of equipartition of energy between the two species in a plasma can be obtained. We study this rate for the asymptotic cases given in the previous section.

For relativistic ions and nonrelativistic electrons we find, using Eq. (6.2),

$$\frac{d\mathcal{E}_i}{dt} = -\frac{4\pi(Zze^2)^2\rho\ln\Lambda}{mc}\left(1 - \frac{T_e}{T_i}\right), \qquad (7.1)$$

where the subscripts i and e refer to ion and electron. The ion energy $\mathcal{E}_i$ is given by

$$\mathcal{E}_i = \int E_1 4\pi f(p)p^2\,dp\,. \qquad (7.2)$$

The integral in Eq. (7.2) can be done exactly with the result

$$\mathcal{E}_i = \frac{kT_i}{8}(a_i)^2\frac{K_4(a_i) - K_0(a_i)}{K_2(a_i)}, \qquad (7.3)$$

where $a_i = Mc^2/kT_i$. Using Eq. (6.12), we then have

$$\mathcal{E}_i \simeq 3kT_i\,. \qquad (7.4)$$

Equation (7.4) is the expected result for a zero-rest-mass particle. Equation (7.1) can now be integrated for the case $T_i \gtrsim T_e$ and the temperature T_i is found to decay exponentially to the value T_e with the characteristic equipartition relaxation time

$$\tau = 3mckT_e/4\pi(Zze^2)^2\rho\ln\Lambda\,. \qquad (7.5)$$

For nonrelativistic ions and relativistic electrons we find immediately from Eq. (6.5a)

$$\frac{d\mathcal{E}_i}{dt} = -\frac{4\pi(Zze^2)^2\rho\ln\Lambda}{Mc}\left(\frac{T_i - T_e}{T_e}\right), \qquad (7.6)$$

where now $\mathcal{E}_i = \frac{3}{2}kT_i$. Equation (7.6) can be integrated. The ion temperature T_i is now found to decay exponentially to T_e with a characteristic equipartition relaxation time given by

$$\tau = 3MckT_e/8\pi(Zze^2)^2\rho\ln\Lambda\,. \qquad (7.7)$$

For ultrarelativistic ions and electrons, we find directly from Eq. (6.14)

$$\frac{d\mathcal{E}_i}{dt} = -\frac{2\pi(Zze^2)^2\rho c\ln\Lambda}{k}\left(\frac{1}{T_e} - \frac{1}{T_i}\right), \qquad (7.8)$$

where $\mathcal{E}_i = 3kT_i$ and we have used Eqs. (6.13) and (7.4). Equation (7.8) can be readily integrated for $T_i \gtrsim T_e$. The ion temperature T_i decays in this case to the electron temperature T_e with the characteristic equipartition relaxation time

$$\tau = 3(kT_e)^2/2\pi(Zze^2)^2\rho c\ln\Lambda\,. \qquad (7.9)$$

VIII. DISCUSSION

The formulation in this paper of the relativistic energy loss has been given in such a manner as to display as clearly as possible the major differences between a relativistic and nonrelativistic treatment. There were several problems that arose in our study which we now discuss.

The minimum cutoff angle corresponding to a maximum impact parameter presents some problems. The precise nature of the screening cloud about a relativistic particle is obscure. Thus the choise of b_{max} as the Debye length requires the appropriate relativistic generalization. This is an open problem which requires investigation. We leave b_{max} unspecified at present.[15]

The correct velocity dependence of θ_0, the cutoff angle, creates formidable analytic problems. We note that our Eq. (4.6) gives the required velocity dependence when transformed back into l.s. variables. The averaging over orientation angles between $\vec{w}$ and $\vec{v}$ is very difficult when the correct $\ln\Lambda$ dependence is included. We have not been able to solve this problem and have treated $\ln\Lambda$ as a constant; even the choice of an appropriate approximation to Λ is difficult. In Ref. 5 it was a simple matter to give the average values of $|\vec{v} - \vec{w}|^2$, but here the velocity dependence of Λ is so complicated that even an appropriate average value of Λ has so far not been realized. In certain of the asymptotic limits discussed in Sec. VI, a reasonable guess at the appropriate velocity dependence in Λ can occasionally be made.[15,16] The entire question and detailed study of the corresponding dominant and nondominant terms[5] resulting from the use of the full velocity dependence of $\ln\Lambda$ is open and we hope to return to this problem in the future.

The nature of the relativistic interaction in a plasma needs a great deal of study. In our derivation of the cross section it was assumed that the particles, moving with effectively constant velocity in the c.m.s., interact via the Lienard-Wiechert potentials. It was expedient, however, to calculate the cross section in the impulse approximation. We made plausible the nature of the correct angle dependence of the cross section bv comparing our result with a few known limiting cases. The choice of a different cross section is now intimately tied up with the solution of the interaction problem.

Once we leave the Lienard-Wiechert approximation we enter the subject of fully covariant relativistic dynamics of interacting particles. This subject is notoriously difficult and is one of the unsolved problems of modern physics. Since relativistic statistical mechanics relies entirely on relativistic dynamics, the entire subject of relativistic statistical mechanics becomes even more difficult to formulate, most of all to solve. Excellent review articles on these subjects have been presented by Havas[17] and by Prigogine.[18]

In addition to these questions, and by analogy to the nonrelativistic case, there is the problem of a justification for the binary-collision approximation to relativistic Coulomb scattering. This requires, by direct analogy to Refs. 11 and 12, a detailed knowledge of the relativistic Boltzmann and Fokker-Planck equations. Israel[19] has formulated a general relativistic Boltzmann equation. The relativistic Fokker-Planck equation for a plasma in the Lienard-Wiechert approximation has been given by Beliaev and Budker.[20] This relativistic kinetic equation has also been derived from the standpoint of the Prigogine[18] formalism by Mangeney.[21] The relativistic Boltzmann equation for a Coulomb plasma has been given by Akama.[22] Because of the presence of a $\ln\Lambda$ factor in the relativistic Coulomb problem, it would appear that the same sort of stochastic analysis and justification should occur here as in the nonrelativistic situation. However, as far as we know, no such detailed analysis has been forthcoming.

In the papers of Beliaev and Budker[20] and of Akama,[22] their relativistic kinetic equations are used to solve for the rate of energy loss in certain special situations. Where possible, it appears that agreement with our results is obtained. A detailed study of the relativistic Fokker-Planck equation for a test particle in a relativistic plasma should agree with the precise results obtained in this paper. Such a study, along with a calculation of the appropriate Fokker-Planck coefficients, would be most valuable.[23] It would help to clarify many of the similar questions to those which have been aired and partially resolved (e.g., Ref. 11) in the nonrelativistic context.

As in the nonrelativistic case, there is also an additional energy-loss mechanism beside that studied in this paper. This is the loss due to collective effects. A detailed study of the rate of energy loss in exciting collective oscillations in a relativistic plasma has been given by Prentice.[24]

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[15]For a nonrelativistic plasma $b_{\max}=\lambda_D=v_T/\omega_p$, where $\omega_p^2=4\pi\rho e^2/m$ and $v_T^2\simeq kT/m$; for a relativistic plasma we might therefore expect something like $\lambda_D=c/\omega_p$. However, a detailed study of the energy dependence of the cutoff $b_{\max}$ is required. After transforming Eq. (4.6) for θ_0 back into l.s. values for the velocities, a heuristic approximation would be to take $|\vec{w}|^2\simeq\langle|\vec{w}|^2\rangle\simeq kT/m$ for a nonrelativistic plasma and to take $|\vec{w}|^2\simeq\langle|\vec{w}|^2\rangle\simeq c^2$ for a relativistic plasma. The heuristic treatment, however, neglects the detailed velocity and orientation angle dependence of $\ln\Lambda$ which requires further detailed study as discussed in Secs. V and VIII.

[16]Though we have left $\ln\Lambda$ as an unspecified constant throughout, a specific value can be deduced in all cases from a detailed evaluation of $\Lambda\simeq 2/\theta_0$ from Eq. (4.6), for the specific case in question, along with $b=b_{\max}$ given in Ref. 15. We here give a specific accurate estimate for $\ln\Lambda$ in each of the four asymptotic cases of physical interest discussed in the paper: (i) when both the test particle and the field particles are nonrelativistic, a detailed study of Eq. (4.6) for a typical nonrelativistic plasma gives $\ln\Lambda\simeq 15$ as discussed thoroughly in Ref. 5; (ii) when the test particle is relativistic and the field particles are either relativistic or nonrelativistic, we find in both cases from Eq. (4.6) that $\Lambda\simeq E_1'b/Zze^2$. So, for a typical case, we take $Z=z=1$, $b_{\max}\simeq 10^{-2}$ cm and the test particle to be a relativistic electron with $E_1'\simeq 10$ MeV we find $\ln\Lambda\simeq 27$; (iii) when the test particle is nonrelativistic and the field particles are relativistic, we find from Eq. (4.6) that $\Lambda\simeq E_1'v'b/cZze^2$. So, for a typical case where we take $Z=z=1, b_{\max}\simeq 10^{-2}$ cm and the test particle to be a nonrelativistic proton with $E_1'\simeq 3$ MeV, we find $\ln\Lambda\simeq 28$. If instead we take a nonrelativistic electron with $E_1'\simeq 10$ KeV for the test particle, then we find $\ln\Lambda\simeq 20$. Thus in all cases the dominant contribution, with $\ln\Lambda$ appropriately evaluated, accounts for better than 90% of the energy loss.

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